

POISSON MEASURE AS A SPECTRAL MEASURE OF A FAMILY OF COMMUTING SELFADJOINT OPERATORS, CONNECTED WITH SOME MOMENT PROBLEM

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To the memory of my dear daughter Natasha. She all the time stands before my eyes.

ABSTRACT. It is proved that the Poisson measure is a spectral measure of some family of commuting selfadjoint operators acting on a space constructed from some generalization of the moment problem.

1. INTRODUCTION

In the years 1991–1998 in the works [2, 7, 13, 3] the authors have constructed a spectral theory of Jacobi fields, which was a generalization of the spectral theory of one selfadjoint operator generated by one Jacobi matrix to families of commuting selfadjoint operators. In particular, it was shown that the Poisson measure on a definite infinite-dimensional space can be considered as a spectral measure for some Jacobi field [4]. This result is deeply connected with article [11], see also recent work [14].

This article also is connected with a study of a Poisson measure as a spectral measure, but from a different position. In the theory of point probability processes, a procedure of constructing of Poisson measure from Lebesgue-Poisson by means of a Kolmogorov-type extension theorem, see [16], is well-known. The Lebesgue-Poisson measure is given on the space $\Gamma_0(X)$ of finite configurations of points of the space X , but the Poisson measure will be a measure on the space of infinite configurations $\Gamma(X)$.

For the space of functions on $\Gamma_0(X)$, it is possible to define the Kondratiev-Kuna convolution \star , see [12], which is a wide generalization of the classical convolution on sequences of numbers in the theory of classical moment problem. The convolution \star generates, as in classical power case, in a natural way, a system of commuting operators on the Hilbert space constructed by means of a given “moment” sequence. It is found that the Poisson measure will be a spectral measure for this family of operators for the corresponding moment sequences on the space, which is wider than $\Gamma(X)$.

It is necessary to explain what measure π on the space \mathcal{D}' is a Poisson measure in our understanding. We assume that for $\omega \in \mathcal{D}'$ and finite smooth functions f on X there is a “pairing”, $\langle \omega, f \rangle$. Then our measure π is a Poisson measure if and only if for its Laplace transform we have the identity

$$(1.1) \quad \int_{\mathcal{D}'} e^{\langle \omega, f \rangle} d\pi(\omega) = \exp \left(\int_X (e^{f(x)} - 1) d\sigma(x) \right),$$

where σ is some fixed initial (“intensity”) measure on X and f is arbitrary.

The article consists of two sections, — Section 2 and Section 3. In Section 2 we present some results about configurations and a classical account of introducing a Poisson measure. This Section is connected with the works [16, 1, 12, 15, 10] and, of course, with

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[9, 8, 5]. Unfortunately, I can not find books or articles that would contain needed facts about measures and, therefore, it was necessary to write Section 2.

In Section 3 we introduce a Poisson measure as a spectral measure ρ . We explain that in this case $\rho(\Gamma_0(X)) > 0$.

Therefore, if we understand a Poisson measure as a measure π on \mathcal{D}' with the Laplace transform satisfying (1.1), then it is possible that $\rho = \pi$ satisfies the condition $\rho(\Gamma_0(X)) > 0$ and it is natural to interpret the set $\Gamma_0(X)$ as a part of \mathcal{D}' . This remark is connected with the work [9, p. 12]. For more details, see the end of Section 3.

It is necessary to make one refinement to the article [9]. Namely, the definition of “non-overlapping configurations γ ” (formula (2.4) in [9]) was not explained. The results of article [9] are true, using the notion of the set Γ of usual configurations (see (2.2) in Section 2 of the present article).

2. POISSON MEASURE. CLASSICAL ACCOUNT

At first we recall some definitions and notations from the article [9].

Let X be a connected C^∞ non-compact Riemannian manifold. We denote by $\mathcal{D} := C_{\text{fin}}^\infty(X)$ the set of all real-valued infinitely differentiable functions on X with compact support. Denote by $\mathcal{D}_\mathbb{C}$ the complexification of \mathcal{D} . We will consider \mathcal{D} as a nuclear topological space with the projective limit topology. Let $\mathcal{F}_0(\mathcal{D}) := \mathbb{C}$ and $\mathcal{F}_n(\mathcal{D}) := \mathcal{D}_\mathbb{C}^{\widehat{\otimes} n}$, $n \in \mathbb{N}$, i. e., it is the space of all complex-valued symmetric infinitely differentiable functions on X^n with compact supports and corresponding to the $\mathcal{D}_\mathbb{C}^{\widehat{\otimes} n}$ topology (the topology of test functions in theory of generalized functions of variables from X).

Construct the space of finite sequences

$$(2.1) \quad \mathcal{F}_{\text{fin}}(\mathcal{D}) := \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\mathcal{D}) \ni f = (f_0, f_1, \dots), \quad f_n \in \mathcal{F}_n(\mathcal{D}),$$

i. e., sequences f , for which only a finite number of components f_n are different from zero. Convergence in this space is equivalent to uniform finiteness of sequences and coordinate-wise convergence of every coordinate f_n from the space $\mathcal{F}_n(\mathcal{D})$ in the topology mentioned above.

Let us recall the notion of the space $\Gamma = \Gamma(X)$ of all configurations generated by X . It is the set of all locally finite subsets γ of X :

$$(2.2) \quad \Gamma := \left\{ \gamma \subset X \mid |\gamma \cap \Lambda| < \infty \text{ for every compact } \Lambda \subset X \right\}$$

(here $|\cdot|$ is the cardinality of this set). Each $\gamma \in \Gamma$ consists of distinct points from X , and Γ consists of all different configurations γ (subsets of X).

The topology into the space Γ is introduced in the following way. Consider the space \mathcal{D}' of continuous linear functionals on the space \mathcal{D} and the weak topology in \mathcal{D}' (see, e. g., [6], Chap. 1, § 1). Let $\gamma = [x_1, x_2, \dots] \in \Gamma$, $x_1, x_2, \dots \in X$ be a certain configuration. Denote by ω_γ the corresponding generalized function from \mathcal{D}' :

$$(2.3) \quad w_\gamma(\varphi) := \left(\sum_{n=1}^{\infty} \delta_{x_n} \right) (\varphi) = \sum_{n=1}^{\infty} \delta_{x_n}(\varphi) =: \sum_{n=1}^{\infty} \varphi(x_n) =: \langle \gamma, \varphi \rangle \quad (\varphi \in \mathcal{D})$$

(the sum in (2.3) is finite, since φ is a finite function and x_n “tends to infinity”). Thus, we have a one-to-one correspondence $\Gamma \ni \gamma \longleftrightarrow \omega_\gamma \in \mathcal{D}'$, and the weak topology in \mathcal{D}' defines some topology on Γ , known as the vague topology (we use this definition, but usually instead of the space \mathcal{D} we use the space of finite continuous functions with uniformly finite convergence).

Denote the set of all finite configurations $\xi = [x_1, \dots, x_n]$, where $x_1, \dots, x_n \in X$, $x_j \neq x_k$ if $j \neq k$, $n \in \mathbb{N}$, by $\Gamma_0 = \Gamma_0(X) \subset \Gamma(X) = \Gamma$. We will understand Γ_0 as a part

of the space Γ and topologize it with the relative topology of the space Γ (in the second part of this Section we will introduce another topologization of the space Γ_0). We will denote elements of the space Γ by γ, θ, \dots and elements of the space Γ_0 by ξ, η, \dots

It is useful to do the following remarks to the notion of vague topology.

Suppose we have a given topological space T with neighborhoods u, v, \dots and some set $A \subset T$. We introduce a topology into A , using, as a system of neighborhoods, the intersections $u \cap A, v \cap A, \dots$, i. e., we introduce into A the relative topology. Let \overline{A} be the closure of A in T . Then the relative topology in \overline{A} completely defined by the relative topology in A .

In our case we introduce the relative topology into $A = \Gamma_0(X)$, using the complete space $T = \mathcal{D}' \supset \Gamma_0(X)$. As follows from (2.3), $\Gamma_0(X)$ is a dense set in $\Gamma(X) = \overline{A}$ in topology of the space \mathcal{D}' . As a result, the vague topology in $\Gamma_0(X)$ completely defines the vague topology in $\Gamma(X)$.

In other words, in definition (2.3) it is possible to take γ to be only finite configurations, $\gamma = \xi = [x_1, \dots, x_m] \in \Gamma_0(X)$, $m \in \mathbb{N}$. Thus, finite configurations completely define the topology in $\Gamma(X)$.

But the one-point configurations $[x_1]$, $x_1 \in X$, in principle, do not define this topology uniquely; it is defined by finite sums of δ -functions.

It is convenient to introduce the following construction, connected with space $\Gamma_0 = \Gamma_0(X)$. Denote by $\Gamma^{(n)} = \Gamma_X^{(n)}$ the set of all finite configurations using the space X^n , $n \in \mathbb{N}$; $\Gamma^{(0)} = \Gamma_X^{(0)} = \emptyset$.

Namely, we understand, by $\Gamma^{(n)}$ for $n \in \mathbb{N}$, a subset of the space $X^n = (x_1, \dots, x_n)$ of points that are symmetric (i. e. the point (x_1, \dots, x_n) does not depend on the order of $x_1, \dots, x_n \in X$) and their "coordinate $x_j \in X$ " are different.

Then the equality of disjunct summands is obvious,

$$(2.4) \quad \Gamma_0 = \Gamma_0(X) = \bigsqcup_{n=0}^{\infty} \Gamma^{(n)} = \bigsqcup_{n=0}^{\infty} \Gamma_X^{(n)}.$$

Let us return to the definition (2.1) of the space $\mathcal{F}_{\text{fin}}(\mathcal{D})$. Every element of this space is a finite vector of the following type:

$$(2.5) \quad f = (f_0, f_1, \dots), \quad f_n \in \mathcal{F}_n(\mathcal{D}) = \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}, \quad n \in \mathbb{N}; \quad f_0 \in \mathbb{C}.$$

Fix $n \in \mathbb{N}$. The component f_n of this vector may be understood as a smooth complex-valued finite function of point $\xi \in \Gamma^{(n)} \subset X^n$, which is symmetric. Conversely, such function is always a component f_n of a certain vector f (2.5). For $n = 0$ we can understand $f_0 \in \mathbb{C}$ as the value in the point \emptyset of such a function.

Thus, we can understand the vector f from (2.5) as a function

$$(2.6) \quad \Gamma_0 = \bigsqcup_{n=0}^{\infty} \Gamma^{(n)} \ni \xi \mapsto f(\xi) \in \mathbb{C},$$

where, for every $n \in \mathbb{N}$, the values of such a function $\Gamma^{(n)} \ni \xi \mapsto f(\xi)$ are infinitely differentiable finite symmetric function on $X^n \supset \Gamma^{(n)}$. For $n = 0$ we have $\emptyset \mapsto f(\emptyset) \in \mathbb{C}$. Additionally, every function (2.6) must be finite in the "direction" n , i. e., $f(\xi) = 0$, where ξ belongs to $\bigsqcup_{n=m}^{\infty} \Gamma^{(n)}$ for m large enough.

In what follows, we will identify the vector f of type (2.1) with a corresponding finite function $f(\xi)$ on the space (2.6). Below, as a rule, the functions of type (2.6) will be denoted by $f, g, \dots: \Gamma_0 \ni \xi \mapsto f(\xi) \in \mathbb{C}, \dots$

The above account and formulas (2.1)–(2.6) are given for the space X . But we can, instead of X , use its subset $Y \subset X$, topologized with the relative topology. Then we get the objects $\Gamma(Y)$, $\Gamma_0(Y)$, $\Gamma^{(n)}(Y)$, $n \in \mathbb{N}_0$.

In this article, as in [9], the Kondratiev–Kuna convolution \star [12, 15] plays an essential role.

Let us recall the corresponding definitions. Let us take vectors (2.1), (2.5) in the form of functions on Γ_0 , $f : \Gamma_0 \ni \xi \mapsto f(\xi) \in \mathbb{C}$; $f \in \mathcal{F}_{\text{fin}}(\mathcal{D})$. For two such functions f, g we introduce the following convolution:

$$(2.7) \quad (f \star g)(\xi) := \sum_{\xi' \sqcup \xi'' = \xi} f(\xi')g(\xi'') = \sum_{\xi' \sqcup \xi'' \sqcup \xi''' = \xi} f(\xi' \sqcup \xi'')g(\xi'' \sqcup \xi'''),$$

$f, g \in \mathcal{F}_{\text{fin}}(\mathcal{D})$

(all sums in (2.7) are finite). This convolutions turns $\mathcal{F}_{\text{fin}}(\mathcal{D})$ into a commutative algebra with involution $f = f(\xi) \rightarrow \overline{f}(\xi) = \overline{f}$ and identity $e = (1, 0, 0, \dots)$; $e(\emptyset) = 1$ and $e(\xi) = 0$ for other $\xi \in \Gamma_0$. We will denote this algebra by \mathcal{A} : $\mathcal{F}_{\text{fin}}(\mathcal{D}) = \mathcal{A}$.

For the algebra \mathcal{A} it is natural to introduce the notion of a character $\chi_\varphi(\xi)$. We will call a character the following function:

$$(2.8) \quad \chi_\varphi : \Gamma_0 \rightarrow \mathbb{R}, \quad \xi \rightarrow \chi_\varphi(\xi) = \prod_{x \in \xi} \varphi(x), \quad \xi \in \Gamma_0 \setminus \emptyset; \quad \chi_\varphi(\emptyset) = 1,$$

where $\varphi \in \mathcal{D}$ is given. One may easily calculate that

$$(2.9) \quad (\chi_\varphi \star \chi_\psi)(\xi) = \chi_{\varphi + \psi + \varphi\psi}(\xi), \quad \xi \in \Gamma_0; \quad \varphi, \psi \in \mathcal{D}.$$

Let us explain that, in equality (2.9), we have the additional member $\varphi + \psi$, since the algebra \mathcal{A} is an algebra with formally added identity \emptyset in contrast to the ordinary discrete group algebra.

The investigation of convolution \star is deeply connected with the K -transform introduced on the basis of the papers of A. Lenard by Yu. G. Kondratiev and T. Kuna in [12].

This transform is a linear operator, acting from $\mathcal{F}_{\text{fin}}(\mathcal{D})$, which is understood as a space of functions on Γ_0 , into complex-valued functions on $\Gamma \supset \Gamma_0$,

$$(2.10) \quad \mathcal{F}_{\text{fin}}(\mathcal{D}) \ni f \mapsto (Kf)(\gamma) := \sum_{\xi \subset \gamma} f(\xi) = F(\gamma) \in \mathbb{C} \text{ if } \xi \neq \emptyset; \quad F(\emptyset) = f(\emptyset).$$

The sum in (2.10) is finite. We do not repeat the definitions, given in [9], of the spaces between which the operator K acts. Such an information about the spaces under consideration will be given further in this article.

Note that that K -transform has an algebraically inverse operator. Namely, it is easy to prove that if for $f \in \mathcal{F}_{\text{fin}}(\mathcal{D})$,

$$(2.11) \quad (Kf)(\gamma) = F(\gamma), \quad \gamma \in \Gamma,$$

then

$$(K^{-1}F)(\xi) = \sum_{\eta \in \xi} (-1)^{|\xi \setminus \eta|} F(\eta), \quad \xi \in \Gamma_0.$$

The K -transform has also one remarkable property: it maps the algebra \mathcal{A} into an algebra of functions on Γ with ordinary multiplication, i. e.,

$$(2.12) \quad (K(f \star g))(\gamma) = (Kf)(\gamma)(Kg)(\gamma), \quad \gamma \in \Gamma; \quad f, g \in \mathcal{F}_{\text{fin}}(\mathcal{D}) = \mathcal{A}.$$

Let us mention another property of the transform K , — it transfers the functions $f \in \mathcal{F}_{\text{fin}}(\mathcal{D})$ into the functions $(Kf)(\gamma)$, $\gamma \in \Gamma$, that are cylindrical in some sense. Such cylindrical property explains the fact that the inverse operator K^{-1} acts only from a part of values of a function $F(\gamma)$ (namely, only from their values on $\Gamma_0 \subset \Gamma$), see (2.11).

Let us explain this assertion. Every function $f \in \mathcal{F}_{\text{fin}}(\mathcal{D})$ has the form (2.5) and according to (2.1) has a finite number of coordinates, different from zero. In the interpretation of this function f as a function $f(\xi)$ on the space Γ_0 (2.4), this property means

$$(2.13) \quad \bigsqcup_{n=0}^m \Gamma^{(n)} \ni \xi \mapsto f(\xi) \in \mathbb{C}.$$

Thus, every value in the “point” γ of the described function $(Kf)(\gamma)$ by formula (2.10) depends only on a sum of the values $f(\xi)$ with $\xi \subset \gamma$. Consider the case where the set γ consists of infinitely many different points $x \in X$. The sum in (2.10) is finite and does not depend on values of our function $\Gamma_0 \ni \xi \mapsto f(\xi) \in \mathbb{C}$ in points of $\gamma \setminus \xi$. So, the function $(Kf)(\gamma)$ is, in some sense, “almost cylindrical”. Almost, because it is necessary to consider $\gamma = [x_1, x_2, \dots] \in \Gamma$ as a “vector with the coordinates $x \in X$ ”. Thus, the function $(Kf)(\gamma)$, with fixed f , depends only on a finite number of “coordinates $x \in X$ ”.

Lemma 2.1. *Let $f \in \mathcal{F}_{\text{fin}}(\mathcal{D})$. Then the function $(Kf)(\gamma)$, $\gamma \in \Gamma$, is continuous in the vague topology.*

[illegible]

Therefore we have, for fixed $f \in \mathcal{F}_{\text{fin}}(\mathcal{D})$ and fixed $\gamma = [x_1, x_2, \dots] \in \Gamma$,

$$(2.15) \quad \begin{aligned} (Kf)(\gamma) = \sum_{\xi \subset \gamma} f(\xi) &= f(\emptyset) + \sum_{[x_1] \subset \gamma} f([x_1]) + \sum_{[x_1, x_2] \subset \gamma} f([x_1, x_2]) \\ &\quad + \cdots + \sum_{[x_1, \dots, x_n] \subset \gamma} f([x_1, \dots, x_n]) + \dots \end{aligned}$$

Let $f \in \mathcal{F}_{\text{fin}}(\mathcal{D})$ that is understood as a finite vector of form (2.1), $f = (f_0, f_1, \dots)$ with components $f_n \in \Gamma^{(n)}$ (see (2.13)). Then it is possible to calculate, using (2.15), the expression $(Kf)([x_1, \dots, x_n])$, $n \in \mathbb{N}$ (see [12, p. 209]). Namely, denote by $\xi_\alpha = [x_{\alpha_1}, \dots, x_{\alpha_l}] \in \Gamma_0$, where α is the set $\{\alpha_1, \dots, \alpha_l\} =: \alpha$ of all different indexes from $\{1, \dots, n\}$, $1 \leq l \leq n$. Then we have the representation of $(Kf)(\xi)$ as a finite sum,

$$(2.16) \quad (Kf)([x_1, \dots, x_n]) = \sum_{\alpha \subset \{1, \dots, n\}} f(\xi_\alpha), \quad n \in \mathbb{N}.$$

Configuration $\gamma \in \Gamma$ in the function $(Kf)(\gamma)$, (2.15), has the form $\gamma =$

$$[x_1, \dots, x_m, x_{m+1}^0, x_{m+2}^0, \dots]$$

with fixed $m \in \mathbb{N}$ and some fixed points $x_{m+1}^0, x_{m+2}^0, \dots$, tending “to infinity”. Therefore, the mentioned continuity of $(Kf)(\gamma)$ gives continuity of this function in the vague topology. \square

We need once more to turn to the space $\mathcal{F}_{\text{fin}}(\mathcal{D})$. This is a linear complex space with the topology described after definition (2.1). It is useful to know that this space (with its topology) is a projective limit of some Hilbert spaces,

$$(2.17) \quad \mathcal{F}_{\text{fin}}(\mathcal{D}) = \text{pr} \lim_{\tau \in T, p \geq 1} F(H_\tau, p) = \bigcap_{\tau \in T, p \geq 1} F(H_\tau, p).$$

Here $F(H_\tau, p)$ is the weighted Fock space which consists of sequences $f = (f_n)_{n=0}^\infty$, $f_n \in H_{\tau, \mathbb{C}}^{\otimes n} =: F_n(H_\tau)$ such that

$$(2.18) \quad \|f\|_{F(H_\tau, p)}^2 = \sum_{n=0}^{\infty} \|f_n\|_{F_n(H_\tau)}^2 p_n < \infty$$

with the corresponding scalar product. Here $p = (p_n)_{n=0}^\infty$, $p_n \geq 1$ means a number weight, $H_{\tau, \mathbb{C}}$ is the complexification of the Sobolev space $W_2^{\tau_1}(X, \tau_2(x)dm(x))$, where $\tau = (\tau_1, \tau_2(x))$, $\tau_1 \in \mathbb{N}_0$, $\tau_2(x) \geq 1$ is a C^∞ weight and m is Riemannian measure on X .

The space $\mathcal{F}_{\text{fin}}(\mathcal{D})$ (2.1) is a nuclear space; the embedding of the spaces $F(H_\tau, p)$ with corresponding τ and p is of Hilbert-Schmidt type. For these notions, we refer to, e. g., [6].

As we have noted, the space $\mathcal{F}_{\text{fin}}(\mathcal{D})$ is a commutative algebra \mathcal{A} with respect to multiplication \star , is endowed with involution “ $-$ ”, and has a unit element e .

Construct the Hilbert space connected with the algebra $\mathcal{A} = \mathcal{F}_{\text{fin}}(\mathcal{D})$. Namely, consider a linear functional $s \in \mathcal{A}'$, which is called non-negative, if

$$(2.19) \quad s(f \star \bar{f}) \geq 0, \quad f \in \mathcal{A}.$$

Any non-negative functional $s \neq 0$ generates the following quasi-scalar product on \mathcal{A}

$$(2.20) \quad (f, g)_{\mathcal{H}_s} = s(f \star \bar{g}), \quad f, g \in \mathcal{A}.$$

Identifying every $f \in \mathcal{A}$ such that $s(f \star \bar{f}) = 0$ with zero, considering corresponding classes and completing the space of these classes, we construct a Hilbert space \mathcal{H}_s .

In this article we will consider only the case where

$$(2.21) \quad \left\{ f \in \mathcal{A} \mid s(f \star \bar{f}) = 0 \right\} = 0,$$

i. e., the positive (non-degenerate) case. In this case (2.20) is a scalar product on \mathcal{A} and the completion of \mathcal{A} with respect to this scalar product gives our space \mathcal{H}_s .

We finish the first part of this Section, devoted to definitions and facts, which are necessary for following account.

The second part, devoted to the classical account of a Poisson measure, we start with a remark that, for this purpose, it is necessary to change the topology of the space Γ , which we have introduced above by (2.2).

Namely, we have $\Gamma = \Gamma_0 \cup (\Gamma \setminus \Gamma_0)$ and the topology of the part Γ_0 must be another one; the space Γ_0 is represented by (2.4) as a disjoint sum of the spaces $\Gamma^{(n)}$, $n \in \mathbb{N}_0$, each of which is a subspace of $X^n \supset \Gamma^{(n)}$ endowed with the relative topology; $\Gamma^{(0)} = \emptyset$. The convergence in the space Γ_0 (2.4) is that of uniform finiteness and coordinate-wise convergence for every coordinate f_n of a vector $f = (f_0, f_1, \dots) \in \bigsqcup_{n=0}^\infty \Gamma^{(n)}$. We will call this topology on Γ_0 an “ordinary topology”.

The topology of $\Gamma \setminus \Gamma_0$ is the previous vague topology, understanding it as the relative topology on $\Gamma \setminus \Gamma_0 \subset \Gamma$ with the vague topology considered on Γ . Thus, we have to the

end of this Section the representation of Γ as a union of two disjoint topological spaces

$$(2.22) \quad \Gamma = \Gamma_0 \sqcup (\Gamma \setminus \Gamma_0),$$

where Γ_0 is topologized with the stated above ordinary topology and $\Gamma \setminus \Gamma_0$ with the vague topology. We will say in this case that Γ (2.22) is topologized with the ordinary-vague topology.

Let us introduce the classical Lebesgue-Poisson measure. We start with a fixed measure σ on the σ -algebra $\mathcal{B}(X)$ of Borel sets from the space X with a topology given on this space. So we have the measure

$$(2.23) \quad \mathcal{B}(X) \ni \alpha \mapsto \sigma(\alpha) \geq 0.$$

This measure (2.23) must be non-degenerate, i. e., $\sigma(\alpha) > 0$ for every open set $\alpha \subset X$ and non-atomic, i. e., for every $x \in X$ $\sigma(\{x\}) = 0$. Assume that $\sigma(X) = +\infty$. We will call this measure σ initial or an intensity measure.

We fix some $n \in \mathbb{N}$ and denote by $\sigma^{(n)}$ or $\sigma^{\widehat{\otimes} n} = \sigma \times \cdots \times \sigma$ (n times) the symmetric tensor product of the measure σ . This measure $\sigma^{(n)}$ is defined on the σ -algebra of Borel sets $\mathcal{B}(X^n)$ as follows:

$$(2.24) \quad \mathcal{B}(X^n) \ni \alpha^{(n)} \mapsto \sigma^{(n)}(\alpha^{(n)}) := \sigma^{\widehat{\otimes} n}(\alpha^{(n)}) \geq 0.$$

Thus, this measure $\sigma^{(n)}$ is also defined on the sets from space $\Gamma^{(n)} = \Gamma_X^{(n)}$ (see (2.4)), which belong to $\mathcal{B}(X^n)$.

For $n = 0$ we put $\sigma^{(0)}(\emptyset) \geq 0$.

The Lebesgue-Poisson measure $d\lambda(\xi)$ on every set $\alpha \subset \Gamma_0 = \Gamma_0(X)$, which is Borel with respect to the ordinary topology, i. e. $\alpha \in \mathcal{B}(\Gamma_0)$, is defined by the formula

$$(2.25) \quad \begin{aligned} \mathcal{B}(\Gamma_0) = \mathcal{B}(\Gamma_0(X)) \ni \alpha \mapsto \lambda(\alpha) = & \sigma^{(0)}(\emptyset) + \frac{1}{1!} \sigma^{(1)}(\alpha^{(1)}) + \frac{1}{2!} \sigma^{(2)}(\alpha^{(2)}) \\ & + \cdots + \frac{1}{n!} \sigma^{(n)}(\alpha^{(n)}) + \cdots, \end{aligned}$$

where $\alpha^{(n)} = \alpha \cap \Gamma^{(n)}$.

Thus, if we have a function $\Gamma_0 \ni \xi \mapsto f(\xi) \in \mathbb{C}$, which is integrable with respect to $d\lambda(\xi)$, then

$$(2.26) \quad \int_{\Gamma_0} f(\xi) d\lambda(\xi) = f(\emptyset) \sigma^{(0)}(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Gamma^{(n)}} f(x_1, \dots, x_n) d\sigma^{(n)}(x_1, \dots, x_n).$$

In what follows, we set $\sigma^{(0)}(\emptyset) = 1$ in (2.25), (2.26). From (2.25), (2.26) it is easy to conclude that $\lambda(\Gamma_0(Y)) = e^{\sigma(Y)}$, where $Y \subset X$ is a compact subset of X .

After having introduced the Lebesgue-Poisson measure, it is possible to introduce a Poisson measure on the space $\Gamma = \Gamma(X)$. This measure is given on the σ -algebra of Borel sets $\mathcal{B}(\Gamma(X))$ in the ordinary-vague topology.

Recall that this topology on $\Gamma(X)$ is the weak topology in \mathcal{D}' , where $\mathcal{D} = C_{\text{fin}}^{\infty}(X)$ with the standard topology in theory of generalized functions on X . The space $\Gamma(X)$ is included into \mathcal{D}' by $\Gamma(X) \ni \gamma = [x_1, x_2, \dots] \mapsto \sum_{n=1}^{\infty} \delta_{x_n}$, where δ_x denotes the δ -function at the point $x \in X$ (see (2.3)).

Denote by $\mathcal{B}_c(X)$ the sets from $\mathcal{B}(X)$ with compact closures, where $\mathcal{B}(X)$ is the σ -algebra of Borel sets in the topology of X . Let $Y \in \mathcal{B}_c(X)$ be such *fixed* set. Consider the set of configurations $\Gamma(Y)$. By definition (2.2) this set $\Gamma(Y)$ consists only of finite configurations γ , since closure Y in X is compact.

We introduce the vague topology in $\Gamma(Y)$ replacing the space X with Y , i. e., this topology is the weak topology in the space \mathcal{D}'_Y , where $\mathcal{D}_Y = C_{\text{fin}}^{\infty}(Y)$.

It is possible to prove, that this topology is as type of topology in the space (2.4), i. e.,

$$(2.27) \quad \Gamma(Y) = \bigsqcup_{n=0}^{\infty} \Gamma_Y^{(n)} = \Gamma_0(Y),$$

where $\Gamma_Y^{(n)}$ is the set of all configurations of type $[x_1, \dots, x_n]$, where $x_j \in Y$, $1 \leq j \leq n$. In (2.27) we have a disjoint sum of $\Gamma_Y^{(n)}$ with the ordinary topology in each $\Gamma_Y^{(n)}$. Such a result follows from the Lemma 2.5 and its proof can be found in [9].

Thus, it is reasonable to write in (2.27), instead of $\Gamma(Y)$, the symbol $\Gamma_0(Y)$; $Y \in \mathcal{B}_c(X)$.

Note that every function $f \in C_{\text{fin}}^{\infty}(Y)$ can be extended by zero to a function on $C_{\text{fin}}^{\infty}(X)$. So, we have some system of topologies on X , depending on $Y \in \mathcal{B}_c(X)$ (but in this system the set $X \setminus Y$ is considered as one point of the space).

Introduce the “projection” p_Y of a “point” from $\Gamma(X)$ to a “point” from $\Gamma_0(Y)$,

$$(2.28) \quad \Gamma(X) \supset \Gamma_0(Y), \quad \Gamma(X) \ni \gamma \mapsto p_Y \gamma = \gamma_Y := \gamma \bigcap Y \in \Gamma_0(Y);$$

evidently, if $\Gamma_0(Y_1) \supset \Gamma_0(Y_2)$, $Y_2 \subset Y_1$, $Y_1, Y_2 \in \mathcal{B}_c(X)$, then $\Gamma_0(Y_1) \ni \gamma \mapsto p_{Y_2, Y_1} = \gamma_{Y_2} := \gamma \cap Y_2 \in \Gamma_0(Y_2)$; $p_Y = p_{YX}$.

Denote by $p_Y^{-1} \gamma$, $\gamma \in \Gamma_0(X)$, and by $p_{Y_2, Y_1}^{-1} \gamma$, $\gamma \in \Gamma_0(Y_1)$ the full preimage of p_Y and p_{Y_2, Y_1} , defined by (2.28). It is evident that

$$(2.29) \quad \begin{aligned} \Gamma_0(Y) \ni \gamma \mapsto p_Y^{-1} \gamma = \theta \in \Gamma(X) \text{ such that } \theta_Y = \gamma; \\ \Gamma_0(Y_2) \ni \gamma \mapsto p_{Y_2, Y_1}^{-1} \gamma = \theta \in \Gamma_0(Y_1) \text{ such that } \theta_{Y_2} = \gamma. \end{aligned}$$

We will apply the operators A of type p_Y , p_{Y_2, Y_1} , p_Y^{-1} , p_{Y_2, Y_1}^{-1} from (2.28) and (2.29) to different sets of configurations $\gamma \in \alpha$. Of course, such an application $A\alpha$ means that $A\alpha := \{A\gamma, \gamma \in \alpha\}$, i. e., the application is pointwise.

Consider an arbitrary $\Gamma_0(Y)$, where $Y \in \mathcal{B}_c(X)$ with the corresponding ordinary topology in $\Gamma_0(Y)$ and the σ -algebra of Borel sets $\mathcal{B}(\Gamma_0(Y))$ in this topology. We have $\mathcal{B}(\Gamma_0(Y)) \subset \mathcal{B}(\Gamma(X))$, $\mathcal{B}(\Gamma_0(Y_2)) \subset \mathcal{B}(\Gamma_0(Y_1))$ and the relations (2.29) can be rewritten as follows:

$$(2.30) \quad \begin{aligned} \mathcal{B}(\Gamma_0(Y)) \ni \alpha \mapsto \beta \in \mathcal{B}(\Gamma(X)) \text{ such that } p_Y^{-1} \beta =: \beta_Y = \alpha; \\ \mathcal{B}(\Gamma_0(Y_2)) \ni \alpha \mapsto \beta \in \mathcal{B}(\Gamma_0(Y_1)) \text{ such that } p_{Y_2, Y_1}^{-1} \beta =: \beta_{Y_2} = \alpha. \end{aligned}$$

We can construct, due to (2.28), (2.29), (2.30), a projective limit $\text{prlim}_Y \Gamma_0(Y)$ of such spaces with the corresponding σ -algebras $\mathcal{B}(\Gamma_0(Y))$ (see, e. g. [16, 10]).

Now we will define a projective limit on $\Gamma(X)$ of probability measures on $\Gamma_0(Y)$, where $Y \in \mathcal{B}_c(X)$ are arbitrary. Such a family of measures μ^Y is defined on the σ -algebra $\mathcal{B}(\Gamma_0(Y))$; the measure μ^Y must be a probability measure, i. e., $\mu^Y(\Gamma_0(Y)) = 1$, $Y \in \mathcal{B}_c(X)$. Their projective limit, μ^X , is a measure on $\mathcal{B}(\Gamma(X))$ and must also be a probability, $\mu^X(\Gamma(X)) = 1$.

This family of measures μ^Y is supposed to be consistent in the following sense, see (2.30):

$$(2.31) \quad \mu^{Y_2}(\alpha) = \mu^{Y_1}(p_{Y_2, Y_1}^{-1} \alpha), \quad \alpha \in \mathcal{B}(\Gamma_0(Y_2)) \subset \mathcal{B}(\Gamma_0(Y_1))$$

for every $Y_2 \subset Y_1$; $Y_1, Y_2 \in \mathcal{B}_c(X)$; i. e., $\mu^{Y_2}(\beta_{Y_2}) = \mu^{Y_1}(\beta)$, $\beta \in \mathcal{B}(\Gamma_0(Y_1))$.

For the limit measure μ^X in the projective limit, we must also have the following property:

$$(2.32) \quad \mu^Y(\alpha) = \mu^X(p_Y^{-1} \alpha), \quad \alpha \in \mathcal{B}(\Gamma(Y)) \subset \mathcal{B}(\Gamma(X)),$$

$Y \subset X$; $Y \in \mathcal{B}_c(X)$ (see (2.30)); i. e., $\mu^Y(\beta_Y) = \mu^X(\beta)$, $\beta \in \mathcal{B}(\Gamma(X))$.

The question is whether there exists such a projective limit of measures. The answer is given by the corresponding version of a Kolmogorov-type theorem (see [16, 1, 10, 15]).

Theorem 2.2. *Suppose that there exists a consistent family of probability measures μ^Y , $Y \in \mathcal{B}_c(X)$, on the σ -algebras $\mathcal{B}(\Gamma_0(Y))$, i. e., (2.31) is satisfied.*

Then there exists a unique probability measure μ^X on $\mathcal{B}(\Gamma(X))$ such that for every $Y \in \mathcal{B}_c(X)$ we have (2.32).

After these general results about the projective limit of the spaces $\Gamma(Y)$ endowed with corresponding Borel σ -algebras $\mathcal{B}(\Gamma_0(Y))$, $Y \in \mathcal{B}_c(X)$, we can introduce a Poisson measure with the use of the Lebesgue-Poisson measure.

Theorem 2.3. *Consider $Y \in \mathcal{B}_c(X)$, and the probability measures*

$$(2.33) \quad \pi^Y(\alpha) = e^{-\sigma(Y)} \lambda(\alpha), \quad \alpha \in \mathcal{B}(\Gamma_0(Y)),$$

on the σ -algebra $\mathcal{B}(\Gamma_0(Y))$, where $d\lambda(\xi)$ is Lebesgue-Poisson measure (2.25) on $\Gamma_0(Y)$ with the initial measure σ (2.23) given on $\mathcal{B}(X)$.

Using the above mentioned Theorem 2.2 we can conclude that, on $\mathcal{B}(\Gamma(X))$, there exists a unique probability measure π , for which

$$(2.34) \quad \begin{aligned} \pi^Y(\alpha) &= \pi(p_Y^{-1}\alpha), \quad \alpha \in \mathcal{B}(\Gamma(Y)), \quad Y \in \mathcal{B}_c(X); \\ \text{i. e. } \pi^Y(\beta_Y) &= \pi(\beta), \quad \beta \in \mathcal{B}(\Gamma(X)), \end{aligned}$$

where the projection p_Y is given by (2.28), (2.29), (2.30). Such a measure is called a Poisson measure.

Proof. Using (2.25) we conclude that, in our situation, $\lambda(\Gamma_0(Y)) = e^{\sigma(Y)}$, therefore the measure (2.33) is a probability measure, $\pi^Y(\Gamma_0(Y)) = 1$.

The condition (2.31) is also fulfilled. It is necessary to prove that the measure μ^Y of the form $\mu^Y = \pi^Y$, where π^Y is given by (2.33), satisfies equality (2.31). We have

$$\begin{aligned} \mu^{Y_2}(\alpha) &= e^{-\sigma(Y_2)} \lambda(\alpha), \quad \alpha \in \mathcal{B}(\Gamma_0(Y_2)) \subset \mathcal{B}(\Gamma_0(Y_1)), \\ \mu^{Y_1}(p_{Y_2, Y_1}^{-1}\alpha) &= e^{-\sigma(Y_1)} \lambda(p_{Y_2, Y_1}^{-1}\alpha) = e^{-\sigma(Y_1)} \lambda(\beta). \end{aligned}$$

Here $\beta \in \mathcal{B}(\Gamma_0(Y_1))$ is such that $\alpha = \beta_{Y_2} = p_{Y_2, Y_1}\beta$.

Thus, it is necessary to prove that

$$(2.35) \quad e^{-\sigma(Y_1)} \lambda(\beta) = e^{-\sigma(Y_2)} \lambda(\beta_{Y_2}), \quad \beta \in \mathcal{B}(\Gamma_0(Y_1)), \quad Y_2 \subset Y_1 \subset X.$$

The measure σ on X is given by (2.23) and is non-degenerate and non-atomic. The proof of (2.35) in the case of the Lebesgue measure σ on $X = \mathbb{R}^d$ is given in [10]. The general situation is considered in [16, 1]. \square

Consider some properties of the Poisson measure, which will be needed in the sequel.

Lemma 2.4. *The Poisson measure is positive on open sets from $\Gamma(X)$ in the vague topology.*

Proof. Let β be an open set in vague topology from $\Gamma(X)$. It is necessary to prove that $\pi(\beta) > 0$. If $\beta \subset \Gamma(X)$ is an open set, then it is easy to prove that, for every $Y \subset X$, the set $\alpha := \beta_Y = p_Y^{-1}\beta$ is also open in the vague topology, considered on the space $\Gamma(Y)$.

In particular, this is true if $Y \in \mathcal{B}_c(X)$. But using (2.27) we know that the vague topology on $\Gamma(Y) = \Gamma_0(Y)$ is the ordinary topology. Thus, the set α is open in the ordinary topology on $\Gamma_0(Y)$. Using the formula (2.34) we assert that it is necessary to prove that $\pi^Y(\alpha) > 0$, or, using (2.33), that $\lambda(\alpha) > 0$. But this follows from (2.25). \square

Let us mention some simple properties of the Poisson and Lebesgue-Poisson measures introduced by Theorem 2.3 and by definition (2.25).

The Lebesgue-Poisson measure λ is defined on sets $\alpha \subset \Gamma(X)$ which are Borel in the vague topology, $\alpha \in \mathcal{B}(\Gamma(X))$. Let

$$(2.36) \quad X = \bigcup_{n=1}^{\infty} Y_n, \quad Y_1 \subset Y_2 \subset \dots, \quad Y_n \in \mathcal{B}_c(X).$$

Then

$$(2.37) \quad \Gamma_0(X) = \bigcup_{n=1}^{\infty} \Gamma_0(Y_n), \quad \Gamma_0(Y_1) \subset \Gamma_0(Y_2) \subset \dots, \quad \Gamma_0(Y_n) \in \mathcal{B}(\Gamma(X)).$$

Using (2.37) for every $\alpha \in \mathcal{B}(\Gamma(X))$ we get

$$(2.38) \quad \alpha = \bigcup_{n=1}^{\infty} \alpha_n, \quad \text{where} \quad \alpha_n = \alpha \cap \Gamma_0(Y_n) \in \mathcal{B}(\Gamma(X)); \quad \alpha_1 \subset \alpha_2 \subset \dots$$

Absolute additivity of the Poisson measure π , with a use of (2.38) and (2.33), gives

$$(2.39) \quad \pi(\alpha) = \lim_{n \rightarrow \infty} \pi(\alpha_n) = \lim_{n \rightarrow \infty} e^{-\sigma(Y_n)} \lambda(\alpha_n).$$

Lemma 2.5. *The Poisson measure π of the set of all finite configurations is equal to zero, $\pi(\Gamma_0(X)) = 0$.*

Proof. Let $\Lambda \subset X$ be some compact subset of X and $\Gamma(\Lambda)$ a corresponding subset of $\Gamma(X)$, $\Gamma(X) \supset \Gamma(\Lambda)$. All configurations from $\Gamma(\Lambda)$ are finite, therefore we can write (see (2.4))

$$(2.40) \quad \Gamma(\Lambda) = \Gamma_0(\Lambda) = \bigsqcup_{m=1}^{\infty} \Gamma_{\Lambda}^{(m)}.$$

From (2.36) it follows that $\Lambda \subset Y_{n_0}$ for some $n_0 \in \mathbb{N}$. Therefore, $\Gamma_0(\Lambda) \subset \Gamma_0(Y_{n_0}) \subset \Gamma_0(Y_{n_0+1}) \subset \dots$. Take $\alpha = \Gamma_0(\Lambda) \in \mathcal{B}(\Gamma(X))$ in (2.38). Then, in this case, $\alpha_{n_0} = \alpha_{n_0+1} = \alpha_{n_0+2} = \dots$ and (2.39) gives

$$\pi(\Gamma_0(\Lambda)) = \pi(\alpha) = \lim_{n \rightarrow \infty} e^{-\sigma(Y_n)} \lambda(\alpha_n) = \lambda(\alpha_{n_0}) \lim_{n \rightarrow \infty} e^{-\sigma(Y_n)} = 0,$$

since (2.36) takes place, and $\sigma(Y_n) \rightarrow +\infty$.

Therefore, $\pi(\Gamma_0(\Lambda)) = 0$ for every $\Lambda \subset X$, i. e. $\pi(\Gamma_0(X)) = 0$. □

We will also prove two known facts about the measures under consideration.

Theorem 2.6. *The Laplace transform of the Poisson measure $\pi(\alpha)$, $\alpha \in \mathcal{B}(\Gamma(X))$ is given by*

$$(2.41) \quad \int_{\Gamma(X)} e^{\langle \gamma, f \rangle} d\pi(\gamma) = \exp \left(\int_X (e^{f(x)} - 1) d\sigma(x) \right), \quad f \in \mathcal{D}.$$

Proof. For fixed $f \in \mathcal{D}$ we can find a set $Y \in \mathcal{B}_c(X)$ such that $f(x) = 0$, $x \in X \setminus Y$. Using (2.34), (2.33), (2.24) and (2.26) we can write

$$\begin{aligned} \int_{\Gamma(X)} e^{\langle \gamma, f \rangle} d\pi(\gamma) &= \int_{\Gamma(Y)} e^{\langle \gamma, f \rangle} d\pi^Y(\gamma) = e^{-\sigma(Y)} \int_{\Gamma(Y)} e^{\langle \gamma, f \rangle} d\rho(\gamma) \\ &= e^{-\sigma(Y)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{Y^n} e^{\sum_{j=1}^n f(x_j)} d\sigma^{(n)}(x_1, \dots, x_n) \\ &= e^{-\sigma(Y)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_Y e^{f(x)} d\sigma(x) \right)^n = \exp \left(\int_X (e^{f(x)} - 1) d\sigma(x) \right). \end{aligned}$$

□

It is usual to say that, if for some measure ρ , it is possible to define its Laplace transform, and for this transform equality (2.41) it follows, that this measure ρ is a Poisson measure.

At last, we will need the following equality that is a special case of Theorem 4.1 from [12, Example 4.1] (see also [15]).

Proposition 2.7. *The following relation between the Lebesgue-Poisson measure $d\lambda(\xi)$ and Poisson measure $d\pi(\gamma)$ holds true:*

$$(2.42) \quad \int_{\Gamma_0(X)} f(\xi) d\lambda(\xi) = \int_{\Gamma(X)} (Kf)(\gamma) d\pi(\gamma), \quad f \in \mathcal{F}_{\text{fin}}(\mathcal{D}).$$

3. POISSON MEASURE AS A SPECTRAL MEASURE OF SOME FAMILY OF COMMUTATING SELFADJOINT OPERATORS

In the first part of Section 2 we had a Hilbert space \mathcal{H}_s which was constructed in the following way.

We have introduced a commutative algebra \mathcal{A} , whose elements are vectors from the space $\mathcal{F}_{\text{fin}}(\mathcal{D})$ (see (2.1), (2.5), (2.6)) and a composition \star defined by (2.7).

On the space $\mathcal{A} = \mathcal{F}_{\text{fin}}(\mathcal{D})$, we consider a linear functional $s \in \mathcal{A}'$, $s \neq 0$, which is non-negative in the sense of (2.19). We consider now only the case when s is positive, i. e., condition (2.21) is fulfilled. Construct the Hilbert space \mathcal{H}_s that is a completion of \mathcal{A} with respect to the scalar product (2.20).

For this space \mathcal{H}_s in the article [9], we considered a family $(A(\varphi))_{\varphi \in \mathcal{D}}$ of unbounded (in general) operators defined by

$$(3.1) \quad \mathcal{H}_s \supset \mathcal{F}_{\text{fin}}(\mathcal{D}) = \mathcal{A} \ni f \mapsto A(\varphi)f = \varphi \star f \in \mathcal{A},$$

where φ is a function from \mathcal{D} (i. e., a real-valued function from $\mathcal{F}_1(\mathcal{D}) = \mathcal{D}$). The closure $\tilde{A}(\varphi)$ of operator (3.1) is well-defined in the space \mathcal{H}_s and is Hermitian.

Such operators $A(\varphi)$, $\varphi \in \mathcal{D}$, were investigated in the article [9] (and earlier in [5, 8]) even in the general case, when the requirement of positivity (2.21) was omitted, and the Hilbert space \mathcal{H}_s consisted of classes of vectors from \mathcal{A} .

Under some conditions every operator $A(\varphi)$, $\varphi \in \mathcal{D}$, is essentially selfadjoint and their set forms a set of commutative selfadjoint operators acting on the space \mathcal{H}_s . In the article [9] (and in [5, 8]) the spectral representation for this family $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$ was considered and some applications of this theory were given.

In this article, we will consider only the case where the functional $s \in \mathcal{A}'$ has the form of an integral. Namely, let s be given by the integral

$$(3.2) \quad s(f) = \int_{\Gamma_0} f(\xi) d\nu(\xi) = \sum_{n=0}^{\infty} \int_{\Gamma^{(n)}} f(\xi) d\nu(\xi), \quad f \in \mathcal{A} = \mathcal{F}_{\text{fin}}(\mathcal{D}),$$

where $d\nu(\xi)$ is some finite measure on the σ -algebra of Borel sets in ordinary topology Γ_0 , given by (2.22). For $f \in \mathcal{F}_{\text{fin}}(\mathcal{D})$ the vectors (2.5) are finite and every function $f \upharpoonright \Gamma^{(n)}$ is a finite smooth function, therefore, the integral (3.2) always exists.

In the articles [5, 8, 9] the following essential fact was proved: if the measure $d\nu(\xi)$ in the representation (3.2) is such, that for every compact $\Lambda \subset X$ there exists a constant $C_\Lambda > 0$ such that

$$(3.3) \quad \nu(\Gamma_\Lambda^{(n)}) \leq C_\Lambda^n, \quad n \in \mathbb{N}_0,$$

then the closures $\tilde{A}(\varphi)$ of the operators $A(\varphi)$ on the space \mathcal{H}_s make a family $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$ of commuting selfadjoint operators (this result is true even when the condition (2.21) is not fulfilled). Let us explain that $\Gamma_\Lambda^{(n)}$ denotes the space $\Gamma_X^{(n)}$ from (2.4) if we replace X with $\Lambda \subset X$.

Let us pass to a study of the case of a Poisson measure. Recall that a non-atomic initial measure $\mathcal{B}(X) \ni \alpha \mapsto \sigma(\alpha) \geq 0$ is given on Borel sets of the space X . Using this measure by rule (2.25) we construct the corresponding Lebesgue-Poisson measure $\lambda(\xi)$ on the σ -algebra of Borel sets $\mathcal{B}(\Gamma_0)$ with respect to the ordinary topology on Γ_0 .

Theorem 3.1. *The functional s of the form (3.2), where $d\nu(\xi) = d\lambda(\xi)$ is a Lebesgue-Poisson measure on Γ_0 , is positive, i. e., condition (2.21) is fulfilled.*

The condition (3.3) for such a functional is also fulfilled.

Proof. Let $f \in \mathcal{A} = \mathcal{F}_{\text{fin}}(\mathcal{D})$ and $s(f \star \bar{f}) = 0$. Then it is necessary to prove that $f = 0$.

Denote $g = f \star \bar{f} \in \mathcal{F}_{\text{fin}}(\mathcal{D})$. Using (2.12) we conclude that

$$(3.4) \quad (Kg)(\gamma) = (K(f \star \bar{f}))(\gamma) = (Kf)(\gamma)(K\bar{f})(\gamma) = |(Kf)(\gamma)|^2, \quad \gamma \in \Gamma.$$

We apply Proposition 2.7 and the corresponding equality (2.42) to a vector $g \in \mathcal{F}_{\text{fin}}(\mathcal{D})$. We get, using (3.2) with the measure $d\lambda(\xi)$ and (3.4) that

$$(3.5) \quad \begin{aligned} s(f \star \bar{f}) = s(g) &= \int_{\Gamma_0} g(\xi) d\lambda(\xi) = \int_{\Gamma} (Kg)(\gamma) d\pi(\gamma) \\ &= \int_{\Gamma} |(Kf)(\gamma)|^2 d\pi(\gamma) = \int_{\Gamma \setminus \Gamma_0} |(Kf)(\gamma)|^2 d\pi(\gamma). \end{aligned}$$

Here $\lambda(\xi)$ is the Lebesgue-Poisson measure on the Borel σ -algebra of the space Γ_0 with the ordinary topology and $d\pi(\gamma)$ is a Poisson measure on the Borel σ -algebra of the space Γ topologized by the ordinary-vague topology (see (2.22)). Since $\pi(\Gamma_0) = 0$, we have (3.5).

Let $f \in \mathcal{F}_{\text{fin}}(\mathcal{D})$ be such that $s(f \star \bar{f}) = 0$. Then we conclude from (3.5) that

$$(3.6) \quad (Kf)(\gamma) = 0$$

for almost all $\gamma \in \Gamma \setminus \Gamma_0$ with respect to a Poisson measure on $\Gamma \setminus \Gamma_0$. According to Lemma 2.1, the function $(Kf)(\gamma)$, $\gamma \in \Gamma$, is continuous with respect to the vague topology. On other hand, the Poisson measure is positive on open sets from Γ in the vague topology (Lemma 2.4). Therefore the equality (3.6) means that $(Kf)(\gamma)$ is equal to zero for every $\gamma \in \Gamma \setminus \Gamma_0$. But the transform K has an algebraically inverse operator K^{-1} (see 2.11), hence $f = 0$.

Pass to the second part of the Theorem. The Lebesgue-Poisson measure on the space Γ_0 is defined by the series (see (2.25))

$$(3.7) \quad \mathcal{B}(\Gamma_0) \ni \alpha \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{(n)}(\alpha^{(n)}) = \lambda(\alpha),$$

where $\mathcal{B}(\Gamma_0)$ is the σ -algebra of Borel sets in the ordinary topology Γ_0 and $\sigma^{(n)}(\alpha^{(n)})$ are values of the symmetric tensor product $m^{\otimes n}$ of the measure m on X on the set $\alpha^{(n)} := \alpha \cap \Gamma^{(n)}$, $n \in \mathbb{N}_0$ (see (2.4)).

For the bounded function $\Gamma_0 \ni \xi \mapsto f(\xi) \in \mathbb{C}$, measurable with respect to $\mathcal{B}(\Gamma_0)$, we have (see (2.26)) that

$$(3.8) \quad \begin{aligned} \int_{\Gamma_0} f(\xi) d\lambda(\xi) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma^{(n)}} f_n(x_1, \dots, x_n) d\sigma^{(n)}(x_1, \dots, x_n), \\ f_n(x_1, \dots, x_n) &= f(\xi) \upharpoonright \Gamma^{(n)}. \end{aligned}$$

Let $\Lambda \subset X$ be an arbitrary compact set. Then, similarly to (2.4), (3.8), we have for a bounded measurable function $f(\xi)$, $\xi \in \Gamma_0(\Lambda)$, that

$$(3.9) \quad \Gamma_0(\Lambda) = \bigsqcup_{n=0}^{\infty} \Gamma_{\Lambda}^{(n)},$$

$$\int_{\Gamma_0(\Lambda)} |f(\xi)| d\lambda(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma_{\Lambda}^{(n)}} |f_n(x_1, \dots, x_n)| d\sigma^{(n)}(x_1, \dots, x_n).$$

In particular, for $f(\xi) = 1$, $\xi \in \Gamma_0(\Lambda)$, we have

$$(3.10) \quad \sum_{n=0}^{\infty} \frac{1}{n!} \lambda(\Gamma_{\Lambda}^{(n)}) = \lambda(\Gamma_0(\Lambda)) < \infty.$$

From (3.10) and (2.25) we easily conclude that there exists a certain constant $C_{\Lambda} > 0$ such that for every $n \in \mathbb{N}_0$, $\lambda(\Gamma_{\Lambda}^{(n)}) < C_{\Lambda}^n$. Thus, the estimate (3.3) is proved. \square

For us it is necessary to repeat some main results of the spectral theory for a family $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$ of commuting selfadjoint operators on the space \mathcal{H}_s , see [9, Theorem 5.3, Condition 3.5] and estimate (3.10).

Proposition 3.2. *Let the conditions (2.21) and (3.3) for the functional s of the form (3.2) be fulfilled. Then the operators $\tilde{A}(\varphi)$ of the family $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$ are selfadjoint in the space \mathcal{H}_s and commuting. This family generates a Fourier transform I of the following form:*

$$(3.11) \quad \mathcal{F}_{\text{fin}}(\mathcal{D}) \ni f = (f_n)_{n=0}^{\infty} \mapsto (If)(\omega) =: \hat{f}(\omega) = (f, P(\omega))_{\mathcal{F}(H)}$$

$$= \sum_{n=0}^{\infty} (f_n, P_n(\omega))_{\mathcal{F}_n(H)} \in L^2(\mathcal{D}', d\rho(\omega)).$$

Here ρ is the spectral measure of the family, being a probability Borel measure on the space \mathcal{D}' of generalized functions ω with weak topology, i. e., on the σ -algebra $\mathcal{B}(\mathcal{D}')$. The closure \tilde{I} by continuity of the operator I is a unitary operator between the spaces \mathcal{H}_s and $L^2(\mathcal{D}', d\rho(\omega))$. It maps each operator $\tilde{A}(\varphi)$ into an operator of multiplication by the function $\langle \omega, \varphi \rangle$.

In (3.11), $P(\omega) = (P_n(\omega))_{n=0}^{\infty}$, where the functions $\mathcal{D}' \ni \omega \mapsto P_n(\omega) \in (\mathcal{D}^{\otimes n})'$, $n \in \mathbb{N}_0$, are similar to polynomials of the first kind in the classical moment problem; $P_0(\omega) = 1$, $\omega \in \mathcal{D}'$. They satisfy the following equality:

$$(3.12) \quad (P(\omega), A(\varphi)f)_{\mathcal{F}(H)} = \langle \omega, \varphi \rangle (P(\omega), f)_{\mathcal{F}(H)}, \quad \varphi \in \mathcal{D}, \quad \omega \in \mathcal{D}', \quad f \in \mathcal{F}_{\text{fin}}(\mathcal{D}).$$

Here $\mathcal{F}(H)$ is the usual symmetric Fock space, constructed from the space

$$H = L^2(X, dm(x)),$$

i. e.,

$$\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(H).$$

The equality (3.12) means that $P(\omega)$ is a joint generalized eigenvector for the family $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$ of the operators $\tilde{A}(\varphi)$ with the eigenvalue $\langle \omega, \varphi \rangle$.

Note that, to prove Proposition 3.2, i. e., Theorem 5.3 from [9], it is necessary to construct some quasi-nuclear rigging of the space \mathcal{H}_s . This rigging is constructed by means of spaces (2.18), for details see in [9, 8, 5] and the book [6].

Let us pass to some results from [9, 8], that are more deeply connected with a Poisson measure as the spectral measure. At first, we will give some results that are stated in the article [8, Theorem 4.1] (see also [9, Section 6]).

Proposition 3.3. *For any $\omega \in \mathcal{D}'$, consider the function*

$$(3.13) \quad e^{\langle \omega, \log(1+\varphi) \rangle},$$

where $\varphi \in \mathcal{D}$ and $\varphi(x) > -1$, $x \in X$. This function can be decomposed into a series in tensor powers $\varphi^{\otimes n}$ in the following way:

$$(3.14) \quad e^{\langle \omega, \log(1+\varphi) \rangle} = \sum_{n=0}^{\infty} (\varphi^{\otimes n}, P_n(\omega))_{\mathcal{F}_n(H)},$$

where the coefficients of this decomposition are just $P_n(\omega)$ from (3.12).

Let $\psi \in \mathcal{D}$ be arbitrary. Then the function $e^{\psi(x)} - 1$ belongs to \mathcal{D} and its values are greater than -1 . Therefore, we can take this function to be $\varphi(x)$ in the expression (3.13). Thus, it is possible to write, for $\psi \in \mathcal{D}$,

$$(3.15) \quad X \ni x \mapsto \varphi(x) = e^{\psi(x)} - 1 \in \mathcal{D}; \quad \varphi(x) > -1, \quad x \in X; \quad \psi \in \mathcal{D} \text{ is arbitrary.}$$

Using this change (3.15) of the function φ to ψ , we can rewrite the equality (3.14) in the form

$$(3.16) \quad e^{\langle \omega, \psi \rangle} = \sum_{n=0}^{\infty} \left((e^{\psi} - 1)^{\otimes n}, P_n(\omega) \right)_{\mathcal{F}_n(H)}, \quad \psi \in \mathcal{D}, \quad \omega \in \mathcal{D}'.$$

Let $\varphi \in \mathcal{D}$ be arbitrary. Recall that we have introduced the notion of a character χ_φ , $\varphi \in \mathcal{D}$, by means of the identity (2.8). This definition is of type (2.6), i. e., we are given some function on Γ_0 . But such a function can be given as a sequence of type (2.5), instead of (2.6). So, we have the following definition of the character χ_φ :

$$(3.17) \quad \begin{aligned} \chi_\varphi(\xi) &= \prod_{x \in \xi} \varphi(x), \quad \xi \in \Gamma_0 \setminus \emptyset; \quad \chi_\varphi(\emptyset) = 1. \\ &\parallel \\ &(1, \varphi, \varphi^{\otimes 2}, \dots, \varphi^{\otimes n}, \dots); \quad \varphi \in \mathcal{D}. \end{aligned}$$

Therefore, the right-hand side of the equality (3.14) can be understood as the right-hand side of the equality (3.11) with $f_n = \varphi^{\otimes n}$. Thus, if we prove that, in the case of the Hilbert space \mathcal{H}_s (constructed from the functional s of the form (3.2) with Lebesgue-Poisson measure $d\nu(\xi) = d\lambda(\xi)$) the vector (3.17) χ_φ belongs to the space \mathcal{H}_s , then it is possible to understand (3.11) as the Fourier transform \tilde{I} of a vector $\chi_\varphi \in \mathcal{H}_s$. We will prove this actually simple fact.

Lemma 3.4. *Let a functional s have the form (3.7) with the Lebesgue-Poisson measure $d\nu(\xi) = d\lambda(\xi)$. Then an arbitrary character χ_φ , $\varphi \in \mathcal{D}$, belongs to the space \mathcal{H}_s .*

Proof. Using the equality (2.9) for the character (3.17) we get

$$(3.18) \quad (\chi_\varphi \star \chi_\varphi)(\xi) = \chi_{2\varphi + \varphi^2}(\xi), \quad \xi \in \Gamma_0; \quad 2\varphi(x) + \varphi^2(x) =: \theta(x), \quad x \in X, \quad \theta \in \mathcal{D}.$$

Denote by $\Lambda \subset X$ the compact set, for which $\theta(x) = 0$, $x \in X \setminus \Lambda$. Then similarly to (3.9) we have with some $c \in (0, \infty)$ that

$$\begin{aligned}
 q(\chi_\varphi) &:= \int_{\Gamma_0} |\chi_\theta(\xi)| d\lambda(\xi) = \int_{\Gamma_0(\Lambda)} |\chi_\theta(\xi)| d\lambda(\xi) \\
 (3.19) \quad &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma_\Lambda^{(n)}} |\theta(x_1), \dots, \theta(x_n)| d\sigma^{(n)}(x_1, \dots, x_n) \\
 &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_{\Lambda} |\theta(x)| d\sigma(x) \right)^n < c < \infty.
 \end{aligned}$$

Introduce the notion of a subcharacter, $\chi_{\varphi, \text{sub}; k}(\xi)$, $k \in \mathbb{N}_0$: instead of (3.17) we put:

$$(3.20) \quad \chi_{\varphi, \text{sub}; k}(\xi) = (1, \varphi, \varphi^{\otimes 2}, \dots, \varphi^{\otimes k}, 0, 0, \dots) \in \mathcal{F}_{\text{fin}}(\mathcal{D}), \quad k \in \mathbb{N}_0.$$

For subcharacters, the formula (3.19) $q(\chi_{\varphi, \text{sub}; k})$ has the form (3.19), but the summation is carried out up to k . Of course,

$$(3.21) \quad \lim_{k \rightarrow \infty} q(\chi_{\varphi, \text{sub}; k}) = q(\chi_\varphi), \quad q(\chi_{\varphi, \text{sub}; k}) \leq q(\chi_{\varphi, \text{sub}; k+1}) \leq c.$$

We have, for $\varphi \in \mathcal{D}$ (see (3.18)), that

$$\begin{aligned}
 (3.22) \quad \|\chi_\varphi\|_{\mathcal{H}_s}^2 &= s(\chi_\varphi \star \overline{\chi_\varphi}) = \int_{\Gamma_0} (\chi_\varphi \star \chi_\varphi) d\lambda(\xi) \\
 &= \int_{\Gamma_0} \chi_\theta(\xi) d\lambda(\xi) \leq \int_{\Gamma_0} |\chi_\theta(\xi)| d\lambda(\xi) = q(\chi_\varphi)
 \end{aligned}$$

and a similar identity for $\chi_{\varphi, \text{sub}; k}(\xi)$. From (3.21), (3.22) we conclude that, in the space \mathcal{H}_s , $\lim_{k \rightarrow \infty} \chi_{\varphi, \text{sub}; k} = \chi_\varphi$. But $\chi_{\varphi, \text{sub}; k} \in \mathcal{F}_{\text{fin}}(\mathcal{D})$, therefore, $\chi_\varphi \in \mathcal{H}_s$. \square

Let us repeat that we will apply Proposition 3.3, the main result of the spectral theory developed in [9, 8, 5, 6]. The corresponding functional s has the form

$$(3.23) \quad s(f) = \int_{\Gamma_0} f(\xi) d\lambda(\xi), \quad f \in \mathcal{A} = \mathcal{F}_{\text{fin}}(\mathcal{D}),$$

where $d\lambda(\xi)$ is the Lebesgue-Poisson measure on Γ_0 .

Consider the series (3.11) (see also (3.15)),

$$\begin{aligned}
 (3.24) \quad \sum_{n=0}^{\infty} ((e^\psi - 1)^{\otimes n}, P_n(\omega))_{\mathcal{F}_n(H)} &= \sum_{n=0}^{\infty} (\varphi^{\otimes n}, P_n(\omega))_{\mathcal{F}_n(H)}, \quad \psi \in \mathcal{D}, \quad \omega \in \mathcal{D}'; \\
 \varphi(x) &= e^{\psi(x)} - 1 \in \mathcal{D}, \quad \varphi(x) > -1, \quad x \in X.
 \end{aligned}$$

As follows from Lemma 3.4, $\chi_\varphi \in H_s$ and, therefore, the series (3.24) can be regarded as the Fourier transform (3.11) $(\tilde{I}\chi_\varphi)(\omega)$ of the function $e^{\langle \omega, \psi \rangle}$ with a fixed $\omega \in \mathcal{D}'$, see (3.16).

Consider the spectral measure $d\rho(\omega)$, $\omega \in \mathcal{D}'$ of the family $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$ of the operators $\tilde{A}(\varphi)$, which is defined on the σ -algebra $\mathcal{B}(\mathcal{D}')$ of Borel sets in the weak topology. Using (3.24) we can write

$$\begin{aligned}
 (3.25) \quad 0 \leq e^{\langle \omega, \psi \rangle} &= \sum_{n=0}^{\infty} (\varphi^{\otimes n}, P_n(\omega))_{\mathcal{F}_n(H)} = (\tilde{I}\chi_\varphi)(\omega), \quad \psi \in \mathcal{D}, \quad \omega \in \mathcal{D}'; \\
 \varphi(x) &= e^{\psi(x)} - 1, \quad x \in X.
 \end{aligned}$$

Integrate the equality (3.25) with respect to $\omega \in \mathcal{D}'$ in measure $d\rho(\omega)$. We get

$$(3.26) \quad \int_{\mathcal{D}'} e^{\langle \omega, \psi \rangle} d\rho(\omega) = \int_{\mathcal{D}'} (\tilde{I}\chi_\varphi)(\omega) d\rho(\omega) \quad \varphi \in \mathcal{D}.$$

Formally, such an integral can be equal to $+\infty$: we integrate a non-negative measurable function (3.25) with respect to a positive finite measure. But we now prove that the integral (3.26) is equal to $s(\chi_\varphi) \in (0, +\infty)$.

It is easy to prove the following general fact.

Lemma 3.5. *Let the conditions (2.19) and positivity (2.21) be fulfilled for the functional $s(f)$, $f \in \mathcal{A} = \mathcal{F}_{\text{fin}}(\mathcal{D})$. Therefore it is possible to introduce the space \mathcal{H}_s . Then the functional s is continuous with respect to the norm of space \mathcal{H}_s and it is possible to extend it to the whole space \mathcal{H}_s .*

Proof. Using the Cauchy-Bunyakovski inequality, for $f, g \in \mathcal{A}$, we can write

$$(3.27) \quad |s(f \star \bar{g})|^2 = |(f, g)_{\mathcal{H}_s}|^2 \leq \|f\|_{\mathcal{H}_s} \|g\|_{\mathcal{H}_s}.$$

Let $g = e$, where e is the unit element of the algebra \mathcal{A} . Then from (3.27) we have

$$|s(f)|^2 = |s(f \star \bar{e})|^2 \leq \|f\|_{\mathcal{H}_s} \|e\|_{\mathcal{H}_s} \leq C \|f\|_{\mathcal{H}_s}, \quad C = \|e\|_{\mathcal{H}_s}.$$

□

Lemma 3.6. *The following equality is true:*

$$(3.28) \quad s(f) = \int_{\mathcal{D}'} (\tilde{I}f)(\omega) d\rho(\omega), \quad f \in \mathcal{H}_s.$$

Proof. Using the Proposition 3.3 we can assert that the operator \tilde{I} is a unitary operator between the spaces \mathcal{H}_s and $L^2(\mathcal{D}', d\rho(\omega))$. Thus

$$(3.29) \quad (f, g)_{\mathcal{H}_s} = \int_{\mathcal{D}'} (\tilde{I}f)(\omega) \overline{(\tilde{I}g)(\omega)} d\rho(\omega), \quad f, g \in \mathcal{H}_s.$$

Let $g = e$ in (3.29), i. e., $e = (1, 0, 0, \dots)$ or, in the form of a function on $\Gamma_0 \ni \xi$, $e(\xi) = 1$ if $\xi = \emptyset$ and 0 for other ξ . We then have, instead of (3.29), that

$$(3.30) \quad (f, e)_{\mathcal{H}_s} = \int_{\mathcal{D}'} (\tilde{I}f)(\omega) d\rho(\omega), \quad f \in \mathcal{H}_s.$$

If $f \in \mathcal{F}_{\text{fin}}(\mathcal{D})$, then $(f, e)_{\mathcal{H}_s} = s(f \star \bar{e}) = s(f)$. But using Lemma 3.5 we can assert that the functional s is continuous also on the space $\mathcal{H}_s \supset \mathcal{F}_{\text{fin}}(\mathcal{D})$. Thus we can write $(f, e)_{\mathcal{H}_s} = s(f)$ also for $f \in \mathcal{H}_s$. Then the equality (3.30) gives (3.28). □

Let us return to the equality (3.26). Since $\forall \varphi \in \mathcal{D}$, the character $\chi_\varphi \in \mathcal{H}_s$ (Lemma 3.4), according to Lemma 3.6 we have the following *essential equality*:

$$(3.31) \quad \int_{\mathcal{D}'} e^{\langle \omega, \psi \rangle} d\rho(\omega) = \int_{\mathcal{D}'} (\tilde{I}\chi_\varphi)(\omega) d\rho(\omega) = s(\chi_\varphi), \quad \varphi \in \mathcal{D}.$$

Theorem 3.7. *Consider the family $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$ of commuting selfadjoint operators $\tilde{A}(\varphi)$ of the form (3.1) on the space \mathcal{H}_s . This space is constructed using the functional s of the form (2.20) and (3.23), where $d\lambda(\xi)$ is a Lebesgue-Poisson measure. These operators $\tilde{A}(\varphi)$ are indeed selfadjoint and commuting.*

The corresponding spectral representation has the form (3.11), where $P(\omega)$ is defined by the equation (3.12). The spectral measure $d\rho(\omega)$ is a non-negative measure on the space \mathcal{D}' and is given on sets of the Borel σ -algebra constructed with respect to the weak topology on \mathcal{D}' .

This measure $d\rho(\omega)$ is Poisson in the following sense: its Laplace transform has the form

$$(3.32) \quad \int_{\mathcal{D}'} e^{\langle \omega, f \rangle} d\rho(\omega) = \exp \left(\int_X (e^{f(x)} - 1) d\sigma(x) \right), \quad f \in \mathcal{D},$$

where $d\sigma(x)$ is the initial measure on X .

Proof. Consider equality (3.25). In this equality ω is a linear functional f from \mathcal{D}' . In particular, we can take ω to be $\gamma \in \Gamma$, where $\gamma = [x_1, x_2, \dots]$, $x_m \in X$ and

$$(3.33) \quad \langle \gamma, \psi \rangle := \omega_\gamma(\psi) = \sum_{m=1}^{\infty} \psi(x_m), \quad \psi \in \mathcal{D}.$$

I. e., we identify, as usual, γ with $\sum_{m=1}^{\infty} \delta_{x_m}$ (see (2.3)).

As a result we have, using (3.33), for the character χ_φ ,

$$(3.34) \quad 0 \leq e^{\langle \gamma, \psi \rangle} = \sum_{n=0}^{\infty} (\varphi^{\otimes n}, P_n(\gamma))_{\mathcal{F}_n(H)} = (\tilde{I}\chi_\varphi)(\gamma),$$

where $\psi \in \mathcal{D}$ and $\varphi(x) = e^{\psi(x)} - 1$, $x \in X$.

Consider in (3.34), instead of χ_φ , the corresponding subcharacter (3.20) $\chi_{\varphi, \text{sub}; k}(\xi) \in \mathcal{F}_{\text{fin}}(\mathcal{D})$. We can write (3.34) in the form

$$(3.35) \quad 0 \leq e^{\langle \gamma, \psi \rangle} = \sum_{n=0}^{\infty} (\varphi^{\otimes n}, P_n(\gamma))_{\mathcal{F}_n(H)} = \lim_{k \rightarrow \infty} \sum_{n=0}^k (\varphi^{\otimes n}, P_n(\gamma))_{\mathcal{F}_n(H)} \\ = \lim_{k \rightarrow \infty} (\tilde{I}\chi_{\varphi, \text{sub}; k})(\gamma) = \lim_{k \rightarrow \infty} (K\chi_{\varphi, \text{sub}; k})(\gamma).$$

We have used in (3.35) the inclusion $\chi_{\varphi, \text{sub}; k} \in \mathcal{F}_{\text{fin}}(\mathcal{D})$ and the following equality for $f \in \mathcal{F}_{\text{fin}}(\mathcal{D})$:

$$(3.36) \quad (\tilde{I}f)(\gamma) = (If)(\gamma) = (Kf)(\gamma)$$

(the equality (3.36) follows from [9, Lemma 6.3]).

Consider the connection (2.42) between the Lebesgue-Poisson measure $d\lambda(\xi)$ and the Poisson measure $d\pi(\gamma)$,

$$(3.37) \quad s(f) = \int_{\Gamma_0} f(\xi) d\lambda(\xi) = \int_{\Gamma} (Kf)(\gamma) d\pi(\gamma),$$

where $f \in \mathcal{F}_{\text{fin}}(\mathcal{D})$.

Integrating (3.36) with respect to $\gamma \in \Gamma$ in the measure $d\pi(\gamma)$ and using (3.37) we get

$$(3.38) \quad \int_{\Gamma} e^{\langle \gamma, \psi \rangle} d\pi(\gamma) = \lim_{k \rightarrow \infty} \int_{\Gamma} (K\chi_{\varphi, \text{sub}; k})(\gamma) d\pi(\gamma) = \lim_{k \rightarrow \infty} s(\chi_{\varphi, \text{sub}; k}) = s(\chi_\varphi).$$

It is easy to prove that one can pass to the limit under the integral. The last limit exists since $\chi_\varphi \in \mathcal{H}_s$ (see Lemma 3.5).

For the classical Poisson measure $d\pi(\gamma)$ we have the general equality (see (2.41)):

$$(3.39) \quad \int_{\Gamma} e^{\langle \gamma, \psi \rangle} d\pi(\gamma) = \exp \left(\int_X (e^{\psi(x)} - 1) d\sigma(x) \right), \quad \psi \in \mathcal{D}.$$

From equalities (3.31), (3.38) and (3.39) we conclude that

$$\int_{\mathcal{D}'} e^{\langle \omega, f \rangle} d\rho(\omega) = \exp \left(\int_X (e^{f(x)} - 1) d\sigma(x) \right), \quad \psi \in \mathcal{D}. \quad \square$$

Let us come back to the question the arose in article [9], at the end of Section 2. It is known, that the Poisson measure π constructed in a classical way from a Lebesgue-Poisson measure by means of a Kolmogorov-type theorem has the following property:

$$(3.40) \quad \pi(\Gamma_0(X)) = 0$$

(see Lemma 2.5).

On the other hand, the spectral measure ρ of the family $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$ of commuting selfadjoint operators can be such that

$$(3.41) \quad \rho(\Gamma_0(X)) > 0,$$

where $\Gamma_0(X)$ is a Borel set in the weak topology of the space \mathcal{D}' . Moreover, property (3.41) is used in the constructions of article [9] (see Theorems 2.10, 6.6), which are essential for Propositions 3.2, 3.3.

We have proved that there is such a family $(\tilde{A}(\varphi))_{\varphi \in \mathcal{D}}$ for which the spectral measure ρ is equal to the Poisson π . Therefore we have some contradiction with (3.41) and (3.40).

But, indeed, this is no contradiction, since our spectral measure is Poissonian only in the sense of Theorem 3.7, i. e., only by definition (1.1).

Let us explain the situation in more details.

Definition (1.1) is based on the assertion that the Laplace transform (1.1) defines uniquely a measure on \mathcal{D}' , when this measure is given on some fixed σ -algebra on \mathcal{D}' (uniquely up to sets of measure zero). But in our case the situation is different. We define π and ρ on $\Gamma(X)$ with a different topology and, therefore, in principle, on different Borel σ -algebras.

Namely in (3.40) the measure π is defined on $\Gamma(X)$ with the ordinary-vague topology (2.22), i. e., on $\Gamma_0(X)$ in the ordinary topology, and on $\Gamma(X) \setminus \Gamma_0(X)$ in the vague topology (i. e., the relative topology as on the part $\Gamma(X) \setminus \Gamma_0(X)$ of $\Gamma(X)$ with the weak topology on $\mathcal{D}' = (C_{\text{fin}}^\infty(X))'$).

In (3.41) we have another topology on $\Gamma(X)$; this is the weak topology on \mathcal{D}' with the inclusion $\Gamma(X) \subset \mathcal{D}'$. Such type of the topology is convenient in spectral theory, see [9], [6].

We can also say that a similar situation about (3.41) and (3.40) is in the work [4], where the spectral measure of special Jacobi fields may be Poissonian in the sense of definition (1.1).

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